Padé/renormalization-group improvement of inclusive semileptonic B decay rates

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Renormalization Group (RG) and optimized Padé-approximant methods are used to estimate the three-loop perturbative contributions to the inclusive semileptonic $b \to u$ and $b \to c$ decay rates. It is noted that the $\overline{MS}$ scheme works favorably in the $b \to u$ case whereas the pole mass scheme shows better convergence in the $b \to c$ case. Upon the inclusion of the estimated three-loop contribution, we find the full perturbative decay rate to be $192\pi^3\Gamma(b \to u\bar{\nu}_\ell\ell^-)/(G_F^2|V_{ub}|^2) = 2065 \pm 290\text{GeV}^5$ and $192\pi^3\Gamma(b \to c\ell^-\bar{\nu}_\ell)/(G_F^2|V_{cb}|^2) = 992 \pm 198\text{GeV}^5$, respectively. The errors are inclusive of theoretical uncertainties and non-perturbative effects. Ultimately, these perturbative contributions reduce the theoretical uncertainty in the extraction of the CKM matrix elements $|V_{ub}|$ and $|V_{cb}|$ from their respective measured inclusive semileptonic branching ratio(s).

The total inclusive rates for semileptonic $b \to c$ and $b \to u$ processes have been calculated to two-loop order in QCD [1,2]:

$$\begin{aligned}\Gamma_{bc}/\kappa &= [m_b]^5\, F\left(\frac{m_c^2}{m_b^2}\right)[1 - 1.67x(\mu) \\ &- (8.9 \pm 0.3 + 3.48L(\mu))x^2(\mu)] \qquad (1)\end{aligned}$$

$$\begin{aligned}\Gamma_{bu}/\kappa &= [m_b(\mu)]^5\,[1 + (4.25360 + 5L(\mu))x(\mu) \\ &+ (26.7848 + 36.9902L(\mu) \\ &+ 17.2917L^2(\mu))x^2(\mu)] \qquad (2)\end{aligned}$$

where

$$\begin{aligned}&x(\mu) \equiv \alpha_s(\mu)/\pi, \quad \kappa \equiv G_F^2|V_{ub}|^2/192\pi^3, \\ &F(r) = 1 - 8r - 12r^2\log(r) + 8r^3 - r^4, \\ &\quad(\text{for } b \to c) : L(\mu) \equiv \log(\mu^2/m_bm_c), \\ &\quad(\text{for } b \to u) : L(\mu) \equiv \log(\mu^2/m_b^2(\mu)). \qquad (3)\end{aligned}$$

In [3] we note that the $b \to u$ rate has less renormalization scale dependence in the $\overline{MS}$ scheme than in the pole-mass scheme. However, in the case of $b \to c$, we find that the total inclusive rate expressed in terms of the b and c pole masses is better behaved [4]. In both cases the scale dependence, which is considerable, provides no optimal choice of renormalization scale μ. Consequently, we estimate the three-loop contributions to the above rate(s) using Padé approximants in order

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to reduce theoretical uncertainties like scale dependence and truncation error. Both decay rates have the following general (perturbative scale-sensitive) form in powers of the strong coupling:

$$S(x) = 1 + R_1 x + R_2 x^2 + R_3 x^3 + \dots \;, \tag{4}$$

where R_1 and R_2 are known as indicated in (1) and (2) . The (unknown) three-loop contribution term R_3 is necessarily of the form:

$$R_3 = c_0 + c_1 L(\mu) + c_2 L^2(\mu) + c_3 L^3(\mu). \tag{5}$$

Since the decay rate is renormalization group (RG) invariant, the following relation holds:

$$\mu^2 \frac{\mathrm{d}\Gamma}{\mathrm{d}\mu^2} = 0. \tag{6}$$

This allows us to evaluate c_1, c_2 and c_3 exactly for both the decay rates. For the case of $b \to c$ we obtain

$$c_1 = -42.4 \pm 1.3 \;,\; c_2 = -7.25 \;,\; c_3 = 0 \tag{7}$$

and for the $b \to u$ case we obtain:

$$c_1 = 249.6,\; c_2 = 178.8 \;,\; c_3 = 50.9 \tag{8}$$

The estimate for the RG-inaccessible coefficient c_0 is obtained from the following estimate developed via asymptotic Padé approximant methods [4]:

$$R_3^{Pade} = \frac{(2+k)R_2^3}{(1+k)R_1^3 + R_1 R_2} \tag{9}$$

The quantity k parametrizes a family of Padé approximants as outlined in [4]. Since both the assumed form of the three-loop contribution and Padé-estimated version are dependent on μ, we proceed by evaluating the following moments:

$$N_j = (j+2) \int_0^1 w^{j+1} R_3(w) \mathrm{d}w \tag{10}$$

where $\log w = -L(\mu)$. Hence, to estimate the value for c_0, we match the scale dependence of the known form (5) to the Padé estimate (9), using the first four moments (N_{-1}, N_0, N_1, N_2) in the perturbative (UV) region. This leads to four linear equations for the four three-loop coefficients $\{c_0, c_1, c_2, c_3\}$. For the $b \to u$ case the method works quite well with $k = 0$ [3]. However, in the case of $b \to c$ decay rate (pole mass scheme) the estimate obtained from $k = 0$ is ill-suited to the series in question [4]. A value of k for the $b \to c$ case can be obtained by finding an optimal k which minimizes the sum of the squares of the relative errors in predicting c_1 and c_2, as given in (7).

Applying the above procedure, we get the following estimated values of c_{0-3} for the $b \to c$ case ($k = -0.94$ for the central value of $b_0 = -8.9$):

$$c_3 = 2.0 \times 10^{-4}, c_2 = -7.68,\; c_1 = -39.7, \quad c_0 = -51.2 \tag{11}$$

The above estimates of the RG-accessible coefficients c_1 and c_2 are within 7% error of their correct values (7).

As a consistency check, we substitute the exact RG values into the moment equations to evaluate c_0. We then find that

$$\begin{aligned} c_0 &= N_{-1} - c_1 - 2c_2 - 6c_3 = -49.4 \;, \\ c_0 &= N_0 - \frac{1}{2}c_1 - \frac{1}{2}c_2 - \frac{3}{4}c_3 = -50.1 \;, \\ c_0 &= N_1 - \frac{1}{3}c_1 - \frac{2}{9}c_2 - \frac{2}{9}c_3 = -50.4 \;, \\ c_0 &= N_2 - \frac{1}{4}c_1 - \frac{1}{8}c_2 - \frac{3}{32}c_3 = -50.6 \;. \end{aligned} \tag{12}$$

which are all within 3% of the Padé-estimated value in Eq.(11).

Similarly, for the $b \to u$ case we have

$$c_3 = 47.61, c_2 = 190.5,\; c_1 = 251.4, \quad c_0 = 206 \tag{13}$$

The RG-accessible coefficients c_1, c_2 and c_3 are all within 7% of their true values (8). A similar consistency check (as above) gets us values of c_0 within 3% of the Padé-estimated value in Eq.(13).

Upon inclusion of the 3-loop contribution we see decreased renormalization scale dependence and the emergence of PMS (Principle of Minimal Sensitivity [5]) extrema for the decay rates [4,3]. For the $b \to c$ case, we find that the PMS value occurs at $\mu = 1$ GeV and yields $\Gamma_{bc}/\kappa = 1047$ GeV^5. We find that this is remarkably close to the FAC (Fastest Apparent Convergence [6]) value which occurs at $\mu = 1.18$ GeV and yields $\Gamma_{bc}/\kappa = 1051$

GeV^5. Similarly, for the $b \to u$ case, we find that the PMS and FAC values occur at $\mu = 1.775$ GeV and 1.835 GeV, respectively. The corresponding reduced rates, are 2069 GeV^5 and 2071 GeV^5, respectively, which are virtually identical.

We take the PMS value of both the decay rates as our central value. Further, we assume that the error in estimating c_0 is the same as our largest error in estimating an RG-accessible coefficient for both cases. We then obtain the following value for the total decay rate for $b \to c$ case:

$$\Gamma_{bc}/\kappa = 992 \pm 198\mathrm{GeV}^5. \tag{14}$$

This error estimate includes the uncertainties in the two-loop contribution (b_0), the b-quark pole mass ($m_b = 4.9 \pm 0.1$ [7]), the strong coupling constant, the three loop estimate, and the uncertainty in estimating non-perturbative effects (for details see [4]). One can extract $|V_{cb}|$ from the above expression with a theoretical error of 10%.

Using the same set of uncertainties (note that $m_b(m_b) = 4.17 \pm 0.05$ [8],[3]) we obtain the following value for the $b \to u$ rate:

$$\Gamma_{bu}/\kappa = 2065 \pm 290\mathrm{GeV}^5 \tag{15}$$

The above expression implies that from the total decay rate one can extract $|V_{ub}|$ with a theoretical error of 7%. However, in this case, the experimental outlook on obtaining a comparably precise measurement of the total inclusive rate (due to the presence of an enormous charm background) does not look promising at present.

We are grateful for support from the Natural Sciences and Engineering Research Council of Canada.

REFERENCES

1. A. Czarnecki and K. Melnikov Phys. Rev. D 59 (1999) 014036.
2. T. van Ritbergen, Phys. Lett. B 454 (1999) 353; T. Seidensticker and M. Steinhauser, Phys. Lett. B 467 (1999) 271.
3. M. R. Ahmady, F. A. Chishtie, V. Elias and T. G. Steele, Phys. Lett. B 479 (2000) 201.
4. M. R. Ahmady et al., Phys. Rev. D 65 (2002) 054021.
5. P. M. Stevenson, Phys. Rev. D 23 (1981) 2916.
6. G. Grunberg, Phys. Lett. B 95 (1980) 70 and Phys. Rev. D 29 (1984) 2315.
7. A.H. Hoang, Phys. Rev. D 61 (2000) 034005 and D 59 (1999) 014039.
8. K. G. Chetyrkin and M. Steinhauser, Phys. Rev. Lett. 83 (1981) 4001.